THE CONVOLUTION OF ARGUMENTATION AND PROVING IN SEARCHING FOR MATHEMATICAL PROOF

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What Proof has been in History?

The concept of mathematical proof has undergone significant changes.

- The Greek concept of ***apodictic proof***, as exemplified by geometrical demonstration in Euclid's *Elements*.
- ***Analytic proof*** (17^{th}-18^{th} c.) became the major characteristic of the European mathematical tradition.
- At the beginning of 20^{th} century, the foundational crisis led to the fundamental distinction between the ***classical*** and the ***constructive*** concepts of mathematical proof and the elaboration of different systems of mathematics (with different associated logical semantics) that make use of either the one or the other concept of ***formal*** proof.

Proof as Process

Joseph Goguen (1941-2006), proposed the concept of *proof-event*, designed to cover all particular exemplifications of proof: apodictic, dialectical, constructive, non-constructive proof, as well as proof steps and computer proofs, incomplete proofs and conjectures.

This concept proved more adequate to study questions of the process of mathematical discovery and demonstration, including the history and the communicative characteristics of the mathematical proving activity, and the role of mathematical communities in the ultimate validation of mathematical proofs.

Goguen's concept of Proof Events

"Mathematicians talk of 'proofs' as real things. But the only things that can actually happen in the real world are proof-events, or provings, which are actual experiences, each occurring at a particular time and place, and involving particular people, who have particular skills as members of an appropriate mathematical community. …

Goguen's concept of Proof Events

A proof-event minimally involves a person having the relevant background and interest, and some mediating physical objects, such as spoken words, gestures, hand written formulae, 3D models, printed words, diagrams, or formulae (we exclude private, purely mental proof-events…). None of these mediating signs can be a "proof" in itself, because it must be interpreted in order to come alive as a proof-event; we will call them *proof objects*. Proof interpretation often requires constructing intermediate proof objects and/or clarifying or correcting existing proof objects. The minimal case of a single prover is perhaps the most common, but it is difficult to study, and moreover, groups of two or more provers discussing proofs are surprisingly common)"

Proof-events as activity of a multi-agent system

We described proof-events as activity of a ***multi-agent system*** (of agents enacting roles of ***prover*** or ***interpreter***, possibly interchangeably) incorporating their history, insofar as they form ***sequences of proof-events*** evolving in time.

Thus, certain temporal aspects of proof-events are modelled using the language of the ***calculus of events*** developed in Kowalski's ***calculus of events***.

Proof-events as activity of a multi-agent system

Using the language of event calculus, we can speak about proof-events and their sequences. The ***calculus of proof-events*** requires a many-sorted predicate logic with equality, with sorts for

- Individual physical objects (humans, chairs, tables, etc.).
- Real numbers, to represent (chronological) time and variable quantities.
- Time-dependent properties, such as states and activities.
- Time-independent propositions, called problems (specified by certain (time-dependent) conditions).

Proof-events as activity of a multi-agent system

- Variable quantities.
- Types of proof-events, whose instantiations mark the beginning and end of time-dependent properties.

The fundamental concepts are the ***proof-events*** and the ***fluents***.

Proof-events as activity of a multi-agent system

Proof-events e take place in space and time (*proving instances/ occurrences*); they refer to a fixed problem (proposition), specified by certain conditions (predicates). A proof-event e has the following internal structure:

$$e \rightleftharpoons \langle present(Intention, Problem), t\rangle$$

which means that an intention (insight or idea or proof sketch, or mathematical argument, etc.) is linguistically articulated for a (time-independent) problem (formulated in the form of a mathematical proposition, specified by certain *conditions*) at time t, which conventionally denotes the time that the communication (*presentation*) has been completed.

Proof-events as activity of a multi-agent system

In this case, we say that the presented output (semiotic "text") is an *exemplification* or an *instance* of a proof-event with respect to the particular fixed problem.

Fluents *f* are sequences of proof-events (proving instances)

$$\{e_n\}_{n=1,2,3,\ldots}$$

evolving in time that refer to a fixed problem.

A fluent is a function that may be interpreted in a model as a set of time points

$$\{t_n\}_{n=1,2,3,\ldots}$$

conventionally denoting the time when the communication output is available.

Extension of Proof-events Calculus

We extend the calculus of proof-events by integrating *argumentation theory* (Toulmin's model, Pollock's concept of *defeasible reasoning*) to represent the relevant stages of a discovery proof-event (incomplete or even false proofs, ideas, valid or invalid inference steps, comments, etc.) in a form of *dialogue* of agents (enacting roles of *prover* or *interpreter*, respectively) that use *arguments* and *counterarguments* or counterexamples in their attempt to clarify the validity of a purported proof.

Proof-events vs. arguments

Arguments and proof-events have three common characteristics:

- a set of premises for a task or problem,
- a method of reasoning, and
- a conclusion.

Proof-events presuppose the existence of at least two agents enacting the roles of ***prover*** or ***interpreter***. Similarly, argumentation involves (at least two) agents enacting the roles of ***supporter*** or ***opponent***.

The layers of communication, understanding, interpretation, and validation that agents use to disseminate their knowledge, are common in both approaches.

Argumentation models

Argumentation models generally contain the following main elements:

an underlying logical language with the definition of the concepts of ***argument***, ***conflict*** between arguments and counterarguments, and ***status*** of argument.

Assuming a multi-agent system, where the agents enact the roles of provers and interpreters, A proof-event e can be understood as a communicated argument $\langle \Phi, c \rangle$

concerning a stated (fixed) problem specified by certain conditions (predicates) and be designated by the pair $e\langle \Phi, c \rangle$

i.e.,

Argumentation models

$$e\langle \Phi, c\rangle \rightleftharpoons \prec e\langle communicate(\text{Problem}, t)\rangle, (communicate\langle \Phi, c\rangle), w \succ$$

where Φ is the *Data* of the argument, c is the *Claim* that refers to a stated (fixed) problem (proposition), specified by certain conditions (predicates) and w are the (possibly implicit) inference rules (*Warrant*) which allow Φ to be logically associated with c, such that:

(i) $\Phi \nvdash \perp$

(ii) $\Phi \vdash c$

(iii) There is no $\Phi' \subset \Phi$, such that $\Phi' \vdash c$.

A counterargument to a proof-event $e\langle \Phi, c\rangle$ represents a new proof-event that can be designated by the pair $e^*\langle \Psi, \beta\rangle$, where Ψ is the Data (generally different from those of Φ) on which is based the Claim (Counterargument) β that refers to the same fixed problem (proposition) stated at time t, specified by the same conditions (predicates).

Argumentation models

Argumentation may require chains (or trees) of reasoning, where claims are used in the assumptions for obtaining further claims [8], so that a proof-event could be an atomic argument or a sequence of arguments (fluent). *Fluents f* are sequences of proof-events (proving instances) evolving in time that refer to a fixed problem, specified by certain conditions Let R be a set of rules of inference. A fluent f is a formula of the form

$$e_1, e_2, e_3 \rightarrow e$$

where $e_1\langle\Phi_1, c_1\rangle, e_2\langle\Phi_2, c_2\rangle, e_3\langle\Phi_3, c_3\rangle$ is a finite, possibly empty, sequence of arguments, such that the conclusion of proof-event e_i is the claim c_i, i.e.

$$conc(e_1) \equiv c_1,\ conc(e_2) \equiv c_2,\ conc(e_3) \equiv c_3$$

for some rule $c_1, c_2, c_3 \rightarrow c \in R$

Argument moves

Arguments can be specified as ***lines of reasons*** leading to a conclusion with consideration of possible counterarguments at each step.

With the explicit construction of the line of reasoning (a chain

$$x_0 x_1, x_2, \ldots x_n$$

where the argument x_i attacks the argument x_{i-1} for $i > 0$) distinct notions of defeat can be conceptualized.

When an agent is in control of the argument, it must select which ***argument move*** to apply. We reserve the term "argument moves" for specific, active tactics that a prover can use to support his claim, relations that indicate links and conflicts at the sequence of proof events.

Argument moves

Given a claim *c* and an argument communicated during the proof-event *e*, possible argument moves, which provide ***support for*** *c* include:

- **Equivalence**: an argument for a claim which is equivalent to (or is) *c*;
- **Elaboration**: an argument for an elaboration of *c*, and

Argument moves which ***oppose*** *c* include:

- **Rebutting**: an argument for a claim which disagrees with *c*;
- **Undercutting**: an argument for a claim which disagrees with a premise of *c*.

Argument moves

Argument moves that support a claim. A proof-event e_1 is *equivalent* with proof-event e_2, if $\Phi = \Phi'$, $c = c'$, although it might be $w \neq w'$, i.e., whenever they have the same data and the same conclusion (although possibly different warrants), i.e.

$$\mathrm{Equivalence}(e, e') \rightleftharpoons e(\Phi, c) = e'(\Phi', c') .$$

Therefore, equivalent proof-events can have different ways of proving

If $e\langle\Phi, c\rangle$ is a proof-event, a set of sentences S is called that *elaborates* or *embellishes* upon e, if the following relation holds

$$\mathrm{Elaboration}(e, S) \rightleftharpoons \mathit{sent}(e) \cap \mathit{sent}(S) \to \mathit{concl}(e) \;\text{ iff }\; \Phi \cup S \vdash c$$

These moves are used for backing our claim and *supporting* our proof, so that

$$\mathrm{Support}(e, t) \rightleftharpoons \mathit{Equivalent}(e, e') \cup \mathit{Elaboration}(e, S)$$

Argument moves

Counterargument moves that attack a claim. A counterargument communicated during the proof-event $e^*\langle\Psi,\beta\rangle$ *attacks* or *rebuts* the conclusion of an argument communicated during the proof-event $e\langle\Phi,c\rangle$, if the following relation holds

$$\text{Rebutting}(e^*,e) \rightleftharpoons \mathit{rebut}(e^*,e) \to \neg \mathit{concl}(e) \ \text{ iff} \vdash \beta \leftrightarrow \neg c$$

A counterargument communicated during the proof-event $e^*\langle\Psi,\beta\rangle$ is called that *undercuts* or *attacks* some of the premises (defeasible inference) of the argument communicated during the proof-event $e\langle\Phi,c\rangle$, if the following relation holds

$$\text{Undercutting}(e^*,e) \rightleftharpoons \mathit{undercut}(e^*,e) \to \neg \mathit{prem}(e) \ \text{ iff} \vdash \beta \leftrightarrow \neg\left(\bigcap_i \Phi_i\right)$$

for $\{\Phi_1,\Phi_2,\ldots,\Phi_n\} \subseteq \Phi$.

Given an argument communicated during the proof-event $e\langle\Phi,c\rangle$, a counterargument communicated during the proof-event $e^*\langle\Psi,\beta\rangle$ *attacks* the argument communicated during the proof-event e, at time t, iff e^* rebuts e or e^* undercuts e. In symbols,

$$\text{Attack}(e^*,t) \rightleftharpoons \mathit{rebut}(e^*,e) \cup \mathit{undercut}(e^*,e)$$

Temporal Predicates

We apply the abovementioned operators combined with the basic temporal predicates from the *calculus of proof-events*.

$Happens(e,t)$ means that a proof-event e occurs at time t.

$Initiates(e,f,t) \rightleftharpoons Happens(e,t_1) \rightarrow \neg attack(e^*,t_1) \cup support(e,t_1)$, at time t_1

which means that if a proof-event e occurs at time t, then there are no counterarguments that attack the validity of the outcome of the proof-event and there is adequate support for our claim at the specific time t_1.

$$Clipped(t_1,f,t_2) \rightleftharpoons \exists e_1, e_1^*, t_1, t[Happens(e,t_1),(t_1 \leq t < t_2) \cap attack(e_1^*,t)]$$
$$\cap[\nexists e_2, t_2(Happens(e_2,t_2) \rightarrow \neg attack(e_1^*,t))], \text{ for } t_1 \leq t < t_2$$

which means that a proof-event clips when there is no proof-event e_2 that attacks the counterargument e_1^* attacking the proof-event e_1 between t_1 and t_2.

Temporal Predicates

$$Terminates(e_1, f, t_1) \rightleftharpoons \exists e, e^*, t_1([attack(e^*, t_1) \rightarrow \neg conc(e) \cup \neg prem(e)] \\ \cap [\nexists e_2, t_2(Happens(e_2, t_2) \rightarrow \neg attack(e^*, t_1))], for\, t_1 < t_2$$

which means that a fluent terminates when there is a counterargument attacking our sequence and there is *no* proof-event e_2 that happens in time t_2, with $t_1 < t_2$, to defend our claim. The termination of a sequence of proof-events may be caused by the proof of the falsity of the problem (there are counter-arguments that attack the conclusion of the proof-event), or the undecidability of the problem (there is a lack of adequate warrants to prove the desideratum).

Temporal Predicates

$$ActiveAt(e, f, t_{n+1}) \leftrightharpoons Happens(t, e_{n+1}, t_{n+1}) \rightarrow \neg attack(e^*_n, t_n) \cup support(e^*_n, t_n), \text{ for every } n \in \mathbb{N}, t_{n+1} > t_n$$

which means that a fluent is active, if there is an argument to support our claim for every counterargument attacking our claim. This means that for every counterargument $e^* \langle \Psi_i, \beta_i \rangle$, $i = 1, \ldots, n$, $n \in \mathbb{N}$ there is a proof-event $e_{n+1}(\Phi_{n+1}, c_{n+1})$, which $Happens(e_{n+1}, t_{n+1})$ and defeats the attack of the counterargument $e^*_n \langle \Psi_n, \beta_n \rangle$, for $t_{n+1} > t_n$.

Levels of argumentation

Following Kakkas and Moraitis, we present three levels of arguments:

- **Object level arguments** represent the possible decisions or actions in a specific domain.
- **First-level priority arguments** express justifications on the object-level arguments in order to resolve possible conflicts.
- **Higher-order priority arguments** are used to deal with potential conflicts between priority arguments of the previous level until all conflicts are resolved.

Levels of argumentation

We can apply the same levels in mathematical proving.

- The *data* and the *claim* of the initial proof-events constitute the **object-level arguments**.
- Proof-events constitute the **first-level priority arguments**, in which we have preferences and justifications in the object-level arguments.
- The proof events that have fulfilled their purpose terminate, while the rest of them continues to the **higher-order priority arguments**. As proof events continue from lower levels to higher, they constitute *fluents*.

Object level arguments

In the object level arguments, we have our claim and the initial representations of arguments. The proof-events that are not attacked constitute the fluent f_0 and continue to the first level priority arguments.

$$Happens(e_i, t_i),\ i = 1, \ldots, m,\ m \in \mathbb{N},\ t_i \le t_m < t$$

$$\forall e_i [Happens(e_i, t_i) \rightarrow \neg attack(e_i^*, t_i) \cap (t_i \le t_m)] \rightarrow Initiates(e_i, f_0, t_m)$$

for $i = 1, \ldots, m,\ m \in \mathbb{N},\ t_i \le t_m < t$.

First-level priority arguments

$$Initiates(e_{m+1}, f_1, t_{m+1}),\ attacks(e^*_{m+1}, f_1, t_{m+1}),$$
$$i = 1, \ldots, m_1,\ m_1 \in \mathbb{N},\ t_{m+1} \leq t_{m+m_1} < t$$

for every $i \in \mathbb{N}$ that we have

$$\begin{aligned}\exists e_{m+i}, e^*_{m+i}, t_{m+i}[attack(e^*_{m+i}, t_{m+i}) \to \neg conc(e_{m+i}) \cup \neg prem(e_{m+i})]\\ \cap \neg prem(e_{m+i})] \cap (t_{m+i} \leq t_{m+m_1} < t)\\ \cap [\nexists e_{m+i+1}, t_{m+i+1}(Happens(e_{m+i+1}, t_{m+i+1})\\ \to \neg attack(e^*_{m+i}, t_{m+i})] \to Terminates(e_{m+i}, f_1, t_{m+m_1})]\end{aligned}$$

so that the proof-events that have been attacked and could not resolve the con-flict, terminate in this fluent.

First-level priority arguments

The rest of them remain active, so we have:

$$ActiveAt(e_{m+j}, f_1, t_{m+m_1}) \text{ for every } j \neq i,\ j \in \mathbb{N}$$

and continues to the second-level priority arguments.

Higher-order priority arguments

If proof-events fail to resolve all the conflicts, our claim cannot be proved and it clips:

$$Clipped(t_i, e, t_n) \text{ at the time } t_n = t_{m(n-1)+mn} \geq t_i$$

If the proof-events manage to deal with all the attacks and

$$\exists j, j \in \mathbb{N}[ActiveAt(e_{m(n-1)+j}, f_n, t_n) \cap \neg Terminates(e, f_n, t_n)]$$
$$\rightarrow Valid(e, t_n), \text{at the time } t_n = t_{m(n-1)+mn} \geq t_i$$

then our claim is proved valid.

A Case Study: Fermat's Last Theorem

Initiation of proof-event: Pierre de Fermat posed the following problem in 1637

> There are no three distinct positive integers a, b, and c other than zero that can satisfy the equation $a^n+b^n=c^n$, if n is an integer greater than two ($n > 2$).

- Fermat claimed to have proved this theorem (uncommunicated proof, therefore it is not a proof-event).
- Leonhard Euler gave a proof of $n = 3$.
- Exponents n=5, 7, 6, 10, 14 were also proved.

A Case Study: Fermat's Last Theorem

- Gabriel Lamé claimed incorrectly that complex numbers can be factored into primes uniquely.
- This gap was indicated by Joseph Liouville.
- ……………………………………………
- The Taniyama–Shimura-Weil conjecture was the method that led to a successful proof of Fermat's Last Theorem.
- Andrew Wiles accomplished a partial proof of this conjecture.
- There was an incorrect critical point in the proof.

A Case Study: Fermat's Last Theorem

- When Wiles was on the verge to quit his attempt, he had an insight that the Kolyvagin–Flach approach and Iwasawa theory were each insufficient on their own, but combined they could be strong enough to overcome this final barrier.
- **Termination of the sequence of proof-events**. In 1994, Wiles submitted two papers that established the modularity theorem for the case of semistable elliptic curves which was the last step in proving Fermat's Last Theorem.

A Case Study: Fermat's Last Theorem

The participation of the agents involved in this sequence of proof-events has the following manifestations:

1. By suggesting partial proofs (for specific cases) of the Theorem.
2. By the rejection of someone else's attempt, pointing out a fault and/or inaccuracy.
3. Through a dialogue between provers in order to detect and resolve weak or insufficiently supported arguments in proving (for instance, Wiles asked his colleagues' contribution, notably Nick Katz and Richard Taylor, when he faced a dead-end in his attempt).

A Case Study: Fermat's Last Theorem

Arguments and counterarguments played a significant role in proving:

- Argumentation is crucial, as it lets counterarguments to be set forth and stronger arguments to survive.
- Both arguments and counterarguments play contribute equally in the construction and justification of the proof.
- The warranted parts of the proofs act as groundwork for the subsequent proofs, while the counterarguments that identify faults in unsuccessful proofs open the way for better-justified proofs and, in some cases, turn the interest of the mathematical community on new unexplored areas.

A Model of FLT proving in terms of the Levels of Argumentation

Object level arguments – Fermat's Conjecture

In the object level arguments, we have Fermat's conjecture as the initial proof-event e_{Fermat} and his claim that he has a proving for this conjecture, without any claim-counterargument e^{*}_{Fermat} clearly opposes this conjecture.

$$Happens(e_{Fermat}, t_{1637}) \cap \neg attack(e^{*}_{Fermat}, t_{1637}) \rightarrow Initiates(e_{Fermat}, f_0, t_{1637})$$

A Model of FLT proving in terms of the Levels of Argumentation

First-level priority arguments - Proofs for specific exponents

In the first-level priority arguments, we have proofs for specific exponents n of the FLT from various mathematicians in different time points.

For the exponent $n=3$, the proof-event $e_{n=3}$ happened when Leonhard Euler gave a proof in 1755. Therefore, we have $Happens(e_{Euler}, t_{1755})$.

Many other well-known mathematicians followed with equivalent proofs that support the validity of the proof for $n=3$. Each prover used a different way (warrant) for proving the conclusion. Thus, their provings are equivalent.

$$Support(e_{n=3}, t_i) \rightarrow Equivallent(e_{n=3}, e_i) \text{ for } i = 1, \ldots, 14 \text{, with}$$

$i=1: (e_{Euler}, t_{1707})$, $i=2: (e_{Kausler}, t_{1802})$, $i=3: (e_{Legendre}, t_{1823})$,

$i=4: (e_{Calzolari}, t_{1855})$, $i=5: (e_{Lamé}, t_{1865})$, $i=6: (e_{Kausler}, t_{1802})$,

$i=7: (e_{Gunther}, t_{1878})$, $i=8: (e_{Gambioli}, t_{1901})$, $i=9: (e_{Krey}, t_{1909})$,

$i=10: (e_{Rycklik}, t_{1910})$, $i=11: (e_{Stockhaus}, t_{1910})$, $i=12: (e_{Carmichael}, t_{1915})$,

$i=13: (e_{Thue}, t_{1917})$, $i=14: (e_{Duarte}, t_{1944})$.

A Model of FLT proving in terms of the Levels of Argumentation

From the aforementioned, we have

$$Happens(e_{Euler}, t_{1755}) \cap Initiates(e_{n=3}, f_1, t_{1755})$$
$$\cap[\neg attack(e^*_{n=3}, t_i) \cup support(e_{n=3}, t_i)] \cap (t_{1755} < t_i)$$
$$\rightarrow ActiveAt(e_{n=3}, f_1, t_i),\ \text{for } t_{1755} < t_i$$

Similarly, we have proofs for $n = 5$ $(e_{n=5})$ and $n = 7$ $(e_{n=7})$ by various mathematicians (provers). The first proof for $n = 5$ belongs to Legendre (1825). Accordingly, we have $Happens(e_{Legendre}, t_{1825})$. Equivalent proofs were also proposed.

$$Support(e_{n=5}, t_j) \rightarrow Equivalent(e_{n=5}, e_j)$$ for $i = 1, \ldots, 10$, with

$j = 1: (e_{Legendre}, t_{1825})$, $j = 2: (e_{Dirichlet}, t_{1825})$, $j = 3: (e_{Gauss}, t_{1875})$,

$j = 4: (e_{Lebergue}, t_{1843})$, $j = 5: (e_{Lamé}, t_{1847})$, $j = 6: (e_{Gambioli}, t_{1901})$,

$j = 7: (e_{Werebrusow}, t_{1905})$, $j = 8: (e_{Rychlik}, t_{1901})$, $j = 9: (e_{Corput}, t_{1159})$

$j = 10: (e_{Terjanian}, t_{1987})$.

A Model of FLT proving in terms of the Levels of Argumentation

From the aforementioned, we have

$$Happens(e_{Legendre}, t_{1825}) \cap Initiates(e_{n=5}, f_1, t_{1825}) \cap [\neg attack(\mathrm{e}^*_{n=5}, t_i) \cup support(e_{n=5}, t_{1825})] \cap (t_{1825} < t_i) \rightarrow ActiveAt(e_{n=5}, f_1, t_i), \text{ for } t_{1825} < t_i.$$

For $n = 7$, the first proof was provided by Lamé in 1839; therefore, we have $Happens(e_{Lamé}, t_{1839})$ and the equivalent supporting provings

$$Support(e_{n=7}, t_k) \rightarrow Equivalent(e_{n=7}, e_k) \text{ for } k = 1, \dots, 10, \text{ with}$$

$k = 1: (e_{Lamé}, t_{1839})$, $k = 2: (e_{Leberguet}, t_{1840})$, $k = 3: (e_{Genocchi}, t_{1876})$,

$k = 4: (e_{Maillet}, t_{1897})$.

Therefore, we have

$$Happens(e_{Lamé}, t_{1839}) \cap Initiates(e_{n=7}, f_1, t_{1839}) \cap [\neg attack(\mathrm{e}^*_{n=7}, t_i) \cup support(e_{n=7}, t_{1839})] \cap (t_{1839} < t_i) \rightarrow ActiveAt(e_{n=7}, f_1, t_i), \text{ for } t_{1839} < t_i.$$

FLT was also proved for the exponents $n = 6, 10, 14$.

A Model of FLT proving in terms of the Levels of Argumentation

Second-level priority arguments – Even exponents

Sophie Germain ($e_{Germain}$) tried unsuccessfully to prove FLT for all even exponents ($e_{n=2p}$), which was proved by Guy Terjanian ($e_{Terjanian}$) in 1977. Germain's attempt was incomplete; thus, it clipped

$$\begin{aligned} Clipped(t_{1776}, e_{n=2p}, t_{1831}) \rightleftharpoons\ & \exists e_{Germain}, e^{*}_{Germain}, t_1 [Happens(e_{Germain}, t_1) \cap \\ & (t_{1776} \leq t_1 < t_{1831}) \cap attack(e^{*}_{Germain}, t)] \cap \\ & [\nexists e_2, t_2 (Happens(e_2, t_2) \rightarrow \neg attack(e^{*}_{Germain}, t_1))],\ for\ t_{1776} \leq t_1 < t_{1831}. \end{aligned}$$

and became active again after the successful proving of Terjanian in 1977.

$$ActiveAt(e_{n=2p}, f_2, t_{1977}) \rightleftharpoons Happens(e_{Terjanian}, t_{1977}) \rightarrow \neg attack(e^{*}_{Terjanian}, t_{1977}).$$

A Model of FLT proving in terms of the Levels of Argumentation

Third-level priority arguments - Ernst Kummer and the theory of ideals

The sequence of proof events continues in the third-level with further attempts for proving FLT. In 1847, Lamé's proof $(e_{Lamé})$ failed, because it incorrectly assumes that complex numbers can be factored into primes uniquely, a gap that was revealed by Liouville $(e^{*}_{Liouville})$. Thus the counterargument generated by Liouville indicated the fault in Lamé's proving and, without adequate proof-events to support $e_{Lamé}$, it was terminated.

$$\begin{aligned}&\exists e_{Lamé}, e^{*}_{Liouville}, t_{1847}[attack(e^{*}_{Liouville}, t_{1847}) \rightarrow \neg conc(e_{Lamé})] \cap (t_{1847} \leq t_1 < t_2)\\ &\qquad \cap [\nexists e_{Lamé}, t_2(Happens(e_{Lamé}, t_2) \rightarrow \neg attack(e^{*}_{Liouville}, t_{1847}))]\\ &\qquad\qquad \rightarrow Terminates(e_{Lamé}, f_3, t_2).\end{aligned}$$

Kummer (e_{Kummer}) proved the conjecture for regular prime numbers $(e_{regular})$, although not for irregular primes $(e_{irregular})$. Therefore, we have

$$ActiveAt(e_{regular}, f_3, t_{1893}) \rightleftharpoons Happens(e_{Kummer}, t_{1893}) \rightarrow \neg attack(e^{*}_{Kummer}, t_{1893}),$$

but

$$\begin{aligned}&\exists e_{Kummer}, e^{*}_{Kummer}, t_{1892}, t_{1893}[attack(e_{Kummer}, t_{1892}) \rightarrow \neg conc(e_{irregular})] \cap\\ &\quad (t_{1892} \leq t_1 < t_{1893}) \cap [\nexists e_{Kummer}, t_2(Happens(e_{Kummer}, t_{1893}) \rightarrow\\ &\quad \neg attack(e^{*}_{Kummer}, t_{1892}))] \rightarrow Terminates(e_{irregular}, f_3, t_{1893}).\end{aligned}$$

A Model of FLT proving in terms of the Levels of Argumentation

Forth-level priority arguments - Connection with elliptic curves

In the forth-level priority, provings are started to connect with elliptic curves. The Taniyama conjecture (e_{TSW}) was proposed in 1955

$$Initiates(e_{TSW}, f_4, t_{1955}) \rightleftharpoons Happens(e_{TSW}, t_{1955}) \rightarrow \neg attack(e^*_{TSW}, t_{1955}) \cup support(e_{TSW}, t_{1955})$$

but it was not proved until 1994, when Andrew Wiles (e_{Wiles}) accomplished a partial proof of this conjecture. Thus we have

$$Happens(e_{Wiles}, t_{1994}) \cap Initiates(e_{TSW}, f_4, t_{1955}) \cap \neg attack(e^*_{TSW}, t_i) \cap (t_{1839} < t_i) \rightarrow ActiveAt(e_{Wiles}, f_4, t_i), \text{ for } t_{1994} < t_i$$

In 1984, Gerhard Frey (e_{Frey}) pointed out a connection between the modularity theorem and Fermat's equation, but FLT still remained unsolved. Thereby, we have $Happens(e_{Frey}, t_{1984})$.

A Model of FLT proving in terms of the Levels of Argumentation

Fifth-level priority arguments – Andrew Wiles

In the fifth-level priority arguments, the procedure and history in the Andrew Wile's attempts is represented. Wiles (e_{Wiles}) discovered and extended an Euler system. He also asked his colleague, Nick Katz, to help him in checking his reasoning for eventual faults.

$$Initiates(e_{Wiles}, f_5, t_{1993}) \rightleftharpoons Happens(e_{Wiles}, t_{1993}) \to \neg attack(e^*_{Wiles}, t_{1993}) \cup support(e_{Katz}, t_{1993}).$$

He presented his work in June 1993, but it soon became evident that there was an incorrect critical point (e^*_{Wiles}) in the proof. Wiles tried for almost a year to resolve this point, firstly by himself and then in collaboration with Richard Taylor (e_{Taylor}), but in vain. Thus, his attempted is clipped on the time period from 1993 until 1994.

$$Clipped(t_{1993}, e_{Wiles}, t_{1994}) \rightleftharpoons \exists e_{Wiles}, e^*_{Wiles}, t_1 [Happens(e_{Wiles}, t_1) \cap (t_{1993} \le t_1 < t_{1994}) \cap attack(e^*_{Wiles}, t_1)] \cap [\nexists e_2, t_2 (Happens(e_{Taylor}, t_2) \to \neg attack(e^*_{Wiles}, t_1))], \ for \ t_{1993} \le t_2 < t_{1993}.$$

In 1994, Wiles managed to overcome this gap by combining Kolyvagin–Flach approach $[Elaboration(e_{Wiles}, S_{Kolyvagin-Flach})]$ and Iwasawa theory $[Elaboration(e_{Wiles}, S_{Iwasawa})]$ and he submitted his final paper which was the last step in proving FLT.

$$ActiveAt(e_{Wiles}, f_5, t_{1994}) \leftrightharpoons Happens(e_{Wiles}, t_{1994}) \to \neg attack(e^*_{Wiles}, t_{1994}) \cap Elaboration(e_{Wiles}, S_{Kolyvagin-Flach}) \cap Elaboration(e_{Wiles}, S_{Iwasawa})$$

A Model of FLT proving in terms of the Levels of Argumentation

Higher-order priority arguments-Fermat's Last Theorem

The proof–event managed to deal with all the attacks and we have

$$[ActiveAt(e_{Wiles}, f_n, t_{1994}) \cap Terminates(e_{Fermat}, f_n, t_{1994})] \rightarrow Valid(e_{Fermat}, t_{1994})$$

at the time t_{1994}.

Thus, FLT is proved valid by Wiles, with the contribution of the other agents that opened the way before him in this ages-long sequence of proof-events.

End of the Proof-event

Thank you for your attention